

12 | Quadratic Equations in One Unknown

Three ways to solve, one number (the discriminant) that predicts the roots, complex numbers when there are no real roots, and the neat link between roots and coefficients.

1. Solving 2. Discriminant 3. Complex Numbers 4. Roots and Coefficients 5. Exam Focus
6. Challenge Zone

1 Solving Quadratic Equations

• CONCEPT Three methods, one goal

A quadratic equation has the form $ax^2 + bx + c = 0$ with $a \neq 0$. Its solutions are called **roots**. Move everything to one side (so the right side is 0) before you start.

- **Factorisation**: quickest when the factors are easy to see. Uses "if $AB = 0$ then $A = 0$ or $B = 0$ ".
- **Completing the square**: always works; it is also how the formula is proved.
- **Quadratic formula**: always works; the safest choice in an exam.

• FORMULA Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In plain words: put a, b, c (with their signs) into the formula. The " $\pm$ " gives the two roots.

◦ WHY IT WORKS Completing the square gives the formula

Divide by a and move the constant: $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Add $\left(\frac{b}{2a}\right)^2$ to both sides:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

Take square roots (remember both $+$ and $-$): $x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$. Subtract $\frac{b}{2a}$ to finish.

• WORKED EXAMPLE Factorisation

Solve $2x^2 - 5x - 3 = 0$.

- $(2x + 1)(x - 3) = 0$, so $x = -\frac{1}{2}$ or $x = 3$.

• WORKED EXAMPLE An equation with a fraction

Solve $\frac{6}{x} = x - 1$.

- Multiply by x ($x \neq 0$): $6 = x^2 - x$, so $x^2 - x - 6 = 0$.
- $(x - 3)(x + 2) = 0$, so $x = 3$ or $x = -2$. Neither is 0, so both are valid.

• **WATCH OUT**

Never divide both sides by an expression containing x . For $x^2 = 3x$, dividing by x loses the root $x = 0$. Write $x^2 - 3x = 0$, then $x(x - 3) = 0$, giving $x = 0$ or 3 .

• **TRY IT**

- Q1** Solve $3x^2 - 7x - 6 = 0$ by factorisation. Basic
- Q2** Solve $x^2 + 6x - 2 = 0$ by completing the square. Leave the answers in surd form. Medium
- Q3** Solve $2x^2 - 3x - 4 = 0$ by the formula. Correct the answers to 3 significant figures. Medium
- Q4** The length of a rectangle is 3 cm more than its width, and its area is 70 cm^2 . Find the width, the length and the perimeter. Medium

2 The Discriminant

• **FORMULA** The discriminant and the nature of the roots

$$\Delta = b^2 - 4ac$$

Δ	Roots of $ax^2 + bx + c = 0$	Graph of $y = ax^2 + bx + c$
$\Delta > 0$	two distinct real roots	cuts the x-axis at 2 points
$\Delta = 0$	two equal real roots	touches the x-axis at 1 point
$\Delta < 0$	no real roots (two complex roots, Stage 3)	does not meet the x-axis

In plain words: Δ is the number under the square root. A positive number has two square roots, zero has one, and a negative number has no real square root.

• **CONCEPT** Using Δ to find a parameter

- Write the equation as $ax^2 + bx + c = 0$ and find Δ in terms of the unknown (say k).
- Set the condition: "equal roots" means $\Delta = 0$; "two distinct real roots" means $\Delta > 0$; "real roots" means $\Delta \geq 0$; "no real roots" means $\Delta < 0$.
- Solve the equation or inequality for k .

A line and a parabola: substitute the line into the parabola to get a quadratic. Then $\Delta > 0, = 0, < 0$ means the line cuts, touches (is a **tangent**) or misses the curve.

• **WORKED EXAMPLE** Equal roots and distinct roots

$$x^2 + (k + 2)x + (k + 5) = 0.$$

- $\Delta = (k + 2)^2 - 4(k + 5) = k^2 - 16$.
- Equal roots: $k^2 - 16 = 0$, so $k = \pm 4$.
- Two distinct real roots: $k^2 > 16$, so $k > 4$ or $k < -4$.

- DSE PRACTICE

- 2013 P2 Q6 Basic

6. Let k be a constant. Solve the equation $(x - k)^2 = 4k^2$.

A. $x = 3k$

B. $x = 5k$

C. $x = -k$ or $x = 3k$

D. $x = -3k$ or $x = 5k$

Working / reason for your choice:

• **WATCH OUT**

"Real roots" is $\Delta \geq 0$ (equal roots are allowed). "Distinct real roots" is $\Delta > 0$. Also, if the coefficient of x^2 contains k , remember that it must not be 0.

• **PRACTICE**

- Q5** Without solving, state the nature of the roots of (a) $2x^2 - 5x + 4 = 0$, (b) $4x^2 - 12x + 9 = 0$, (c) $x^2 + x - 1 = 0$. Basic
- Q6** The equation $kx^2 - 6x + 3 = 0$ has real roots. Find the range of k . Medium
- Q7** The line $y = 2x + k$ and the parabola $y = x^2 - 4x + 7$. DSE style
- (a) Show that the x-coordinates of the intersection points satisfy $x^2 - 6x + (7 - k) = 0$.
- (b) The line is a tangent to the parabola. Find k and the point of contact.
- (c) Find the range of k for which the line cuts the parabola at two distinct points.

3 Complex Numbers

• **CONCEPT** When $\Delta < 0$

There is no real number whose square is -1 , so we **invent** one: the **imaginary unit** i with $i^2 = -1$.

- A **complex number** is $a + bi$, where a, b are real. a is the **real part** and b is the **imaginary part**.
- Two complex numbers are equal only when their real parts are equal **and** their imaginary parts are equal.
- Add, subtract and multiply like ordinary algebra, replacing i^2 by -1 .
- The **conjugate** of $a + bi$ is $a - bi$.
- Powers of i repeat every 4: $i, -1, -i, 1, i, \dots$

• **FORMULA** Conjugate and division

$$(a + bi)(a - bi) = a^2 + b^2 \quad (\text{a real number})$$

In plain words: to divide by a complex number, multiply top and bottom by its conjugate. The bottom becomes real.

• **WHY IT WORKS** Why the product is real, and how the roots appear

Expand: $(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2$, since $i^2 = -1$.

For a real-coefficient quadratic with $\Delta < 0$, the formula gives

$$x = \frac{-b \pm i\sqrt{-\Delta}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{-\Delta}}{2a}i$$

The two roots are **conjugates** of each other.

- DSE PRACTICE

- 2015 P2 Q7 Medium

7. If β is a root of the equation $4x^2 - 5x - 1 = 0$, then $7 + 10\beta - 8\beta^2 =$

A. 5.

B. 7.

C. 9.

D. 11.

Working / reason for your choice:

• **WORKED EXAMPLE Multiply, divide and solve**

- $(2 + i)(3 - 2i) = 6 - 4i + 3i - 2i^2 = 8 - i$.
- $\frac{1 + 2i}{3 - i} = \frac{(1 + 2i)(3 + i)}{(3 - i)(3 + i)} = \frac{1 + 7i}{10} = \frac{1}{10} + \frac{7}{10}i$.
- $x^2 - 4x + 7 = 0$: $\Delta = -12$, so $x = \frac{4 \pm \sqrt{-12}}{2} = 2 \pm \sqrt{3}i$.
- i^{2025} : $2025 = 4 \times 506 + 1$, so $i^{2025} = i$.

• **TRY IT**

Q8 Simplify (a) $(2 + 3i) - (5 - i)$, (b) $(3 + 2i)(1 - 4i)$. Basic

Q9 (a) Simplify $\frac{7 + i}{1 - 2i}$.

(b) Find real numbers x and y such that $(x + 2i)(1 - i) = 5 + yi$. Medium

Q10 Solve $x^2 - 6x + 13 = 0$. Medium

4 Roots and Coefficients

• **FORMULA Sum and product of roots**

If α, β are the roots of $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}$$

and the equation with roots α, β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

In plain words: you can find the sum and product of the roots straight from the coefficients, without solving.

• **WHY IT WORKS Compare two forms**

Since α, β are the roots, $ax^2 + bx + c = a(x - \alpha)(x - \beta) = ax^2 - a(\alpha + \beta)x + a\alpha\beta$. Compare coefficients: $b = -a(\alpha + \beta)$ and $c = a\alpha\beta$. Divide by a .

• **CONCEPT Symmetric expressions**

Most questions ask for an expression in α and β . Rewrite it using $\alpha + \beta$ and $\alpha\beta$:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta, \quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta, \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$$

To build a **new equation**: find the sum S and product P of the new roots, then write $x^2 - Sx + P = 0$.

• **WORKED EXAMPLE New equation from old**

The roots of $2x^2 - 6x + 1 = 0$ are α, β . Then $\alpha + \beta = 3$ and $\alpha\beta = \frac{1}{2}$.

- $\alpha^2 + \beta^2 = 9 - 1 = 8$ and $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{3}{1/2} = 6$.
- Roots $2\alpha, 2\beta$: sum 6, product $4\alpha\beta = 2$, so $x^2 - 6x + 2 = 0$.

• DSE PRACTICE

• 2015 P2 Q34 Medium

34. Let k be a constant. If the roots of the quadratic equation $x^2 + kx - 2 = 0$ are α and β , then $\alpha^2 + \beta^2 =$

A. k^2 .

B. $k^2 + 4$.

C. $k^2 - 4$.

D. $k^2 - 8$.

Working / reason for your choice:

• **KEY IDEA**

This works even when the roots are complex. Also, if a root such as $2 + 3i$ is given for a **real-coefficient** equation, the other root is its conjugate $2 - 3i$.

• **PRACTICE**

Q11 The roots of $2x^2 - 7x + 4 = 0$ are α and β . Medium

(a) Find $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$.

(b) Find a quadratic equation with roots $\alpha + 1$ and $\beta + 1$, with integer coefficients.

Q12 The equation $x^2 - 6x + k = 0$ has roots α and β , where $\beta = 2\alpha$. DSE style

(a) Find α , β and k .

(b) Hence find a quadratic equation with roots α^2 and β^2 .

Q13 $2 + 3i$ is a root of $x^2 + px + q = 0$, where p, q are real. Find p and q . Harder

! Exam Focus

Common DSE question type	Key move
Solve a quadratic; equations reducible to one	Get " $= 0$ " first; use the formula when unsure
Discriminant with a parameter; line and curve	$\Delta = 0, > 0, < 0$ (tangent, cuts, misses)
Complex number arithmetic; solve with i	$i^2 = -1$; multiply by the conjugate to divide
Sum and product of roots; form a new equation	$-\frac{b}{a}$ and $\frac{c}{a}$, then $x^2 - Sx + P = 0$
Word problems (area, distance, age)	Form the quadratic, solve, reject impossible roots

• **WATCH OUT** **Where marks are lost**

- Leaving out " $\pm$ ", or writing b instead of $-b$ in the formula.
- Dividing by an expression with x and losing a root.
- Using $\Delta > 0$ when the question says "real roots" (should be $\Delta \geq 0$).
- Not rejecting a root that is impossible in context (a negative length).
- Leaving i^2 unreplaced; sign errors in a conjugate.
- Forgetting that $\alpha + \beta = -\frac{b}{a}$ has a **minus** sign.

★ Challenge Zone

For students who want more. Some questions go slightly beyond the DSE syllabus.

- DSE PRACTICE

- 2013 P2 Q35 Harder

35. If $\alpha \neq \beta$ and $\begin{cases} 3\alpha = \alpha^2 - 5 \\ 3\beta = \beta^2 - 5 \end{cases}$, then $\alpha\beta =$

A. 3 .

B. -3 .

C. 5 .

D. -5 .

Working / reason for your choice:

• CHALLENGE

- C1 Always real.** Show that $(x - 1)(x - 2) = k^2$ has real roots for every real k .

Hint: expand to the form $ax^2 + bx + c = 0$, find Δ , and show it can never be negative.

- C2 Cubes and ratios.** The roots of $x^2 - 4x + 2 = 0$ are α, β . Find $\alpha^3 + \beta^3$, and find the equation with roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

Hint: $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$.

- C3 Square root of a complex number** (beyond syllabus). Find all complex numbers $z = a + bi$ with $z^2 = 3 + 4i$.

Hint: expand $(a + bi)^2$ and compare the real and imaginary parts to get two equations in a, b .

- C4 Hidden quadratic.** Solve $x^4 - 3x^2 - 4 = 0$, first for real x , then for complex x .

Hint: let $u = x^2$.

Answers

Try it and Practice

Q1 $(3x + 2)(x - 3) = 0$, so $x = -\frac{2}{3}$ or $x = 3$

Q2 $(x + 3)^2 = 11$, so $x = -3 \pm \sqrt{11}$

Q3 $x = \frac{3 \pm \sqrt{9+32}}{4} = \frac{3 \pm \sqrt{41}}{4}$, so $x \approx 2.35$ or $x \approx -0.851$

Q4 width w : $w(w + 3) = 70$, so $w^2 + 3w - 70 = 0$, $(w + 10)(w - 7) = 0$; reject $w = -10$; width 7 cm, length 10 cm, perimeter 34 cm

Q5 (a) $\Delta = 25 - 32 = -7 < 0$: no real roots (b) $\Delta = 144 - 144 = 0$: two equal real roots (c) $\Delta = 1 + 4 = 5 > 0$: two distinct real roots

Q6 $\Delta = 36 - 12k \geq 0$ gives $k \leq 3$, and $k \neq 0$ (otherwise the equation is not quadratic); so $k \leq 3$ and $k \neq 0$

Q7 (a) $x^2 - 4x + 7 = 2x + k$ gives $x^2 - 6x + (7 - k) = 0$ (b) $\Delta = 36 - 4(7 - k) = 8 + 4k = 0$, so $k = -2$; then $x^2 - 6x + 9 = 0$, $x = 3$, $y = 2(3) - 2 = 4$; point of contact $(3, 4)$ (c) $8 + 4k > 0$, so $k > -2$

Q8 (a) $-3 + 4i$ (b) $3 - 12i + 2i - 8i^2 = 11 - 10i$

Q9 (a) $\frac{(7+i)(1+2i)}{5} = \frac{7+14i+i-2}{5} = 1 + 3i$ (b) $(x + 2i)(1 - i) = (x + 2) + (2 - x)i$; so $x + 2 = 5$ and $2 - x = y$, giving $x = 3$, $y = -1$

Q10 $\Delta = 36 - 52 = -16$; $x = \frac{6 \pm 4i}{2} = 3 \pm 2i$

Q11 $\alpha + \beta = \frac{7}{2}$, $\alpha\beta = 2$. (a) $\alpha^2 + \beta^2 = \frac{49}{4} - 4 = \frac{33}{4}$; $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{7/2}{2} = \frac{7}{4}$ (b) sum $= \frac{7}{2} + 2 = \frac{11}{2}$; product $= \alpha\beta + \alpha + \beta + 1 = 2 + \frac{7}{2} + 1 = \frac{13}{2}$; equation $2x^2 - 11x + 13 = 0$

Q12 (a) $\alpha + 2\alpha = 6$, so $\alpha = 2$, $\beta = 4$, $k = \alpha\beta = 8$ (b) sum $= 4 + 16 = 20$, product $= 4 \times 16 = 64$; $x^2 - 20x + 64 = 0$

Q13 the other root is $2 - 3i$; sum $= 4 = -p$, so $p = -4$; product $= 4 + 9 = 13 = q$

Challenge Zone

C1 $x^2 - 3x + (2 - k^2) = 0$; $\Delta = 9 - 4(2 - k^2) = 1 + 4k^2 \geq 1 > 0$ for all real k .

C2 $\alpha + \beta = 4$, $\alpha\beta = 2$. $\alpha^3 + \beta^3 = 64 - 3(2)(4) = 40$. For $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$: sum $= \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{16-4}{2} = 6$, product $= 1$; equation $x^2 - 6x + 1 = 0$.

C3 $a^2 - b^2 = 3$ and $2ab = 4$, so $b = \frac{2}{a}$; $a^2 - \frac{4}{a^2} = 3$, giving $a^4 - 3a^2 - 4 = 0$, $a^2 = 4$; $z = 2 + i$ or $z = -2 - i$.

C4 $(u - 4)(u + 1) = 0$, so $x^2 = 4$ or $x^2 = -1$. Real solutions: $x = \pm 2$. Complex solutions: $x = \pm 2i$ and $x = \pm i$.